Madras College Maths Department
Higher Maths

E&F 1.1 Logarithms and Exponentials

Written solutions for each exercise are available at

http://madrasmaths.com/courses/higher/revison_materials_higher.html

You should check your solutions at the end of each exercise and ask your teacher or attend study support if there any problems.

Logarithms and Exponentials

A logarithm is the inverse* operation of an exponential. It can be used to solve the following equations.

Inverse* function – The proper name for the opposite operation is an inverse operation e.g. the inverse of adding is taking away, the inverse of squaring is square rooting. The formal definition is that f and g are inverses, then $f(g(x))= f(g(x))=x$. We learn about this later in the course

Q) Solve the following:

a) $10^x = 100$

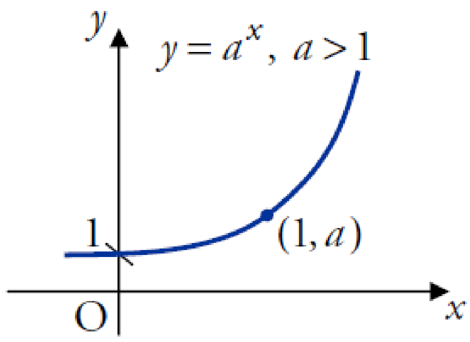
b) $10^x = 50$

c) $7^x = 300$

Exponentials

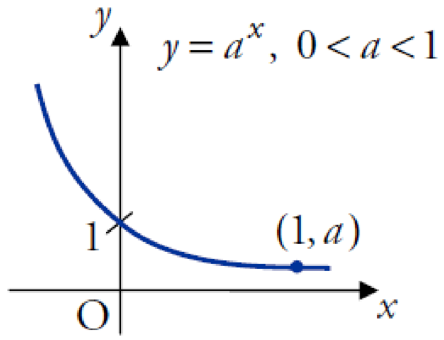
A function of the form $f(x)=a^x$ where $a, x \in \mathbb{R}$ and $a>0$ is known as an **exponential function** to the base a.

If $a>1$ then the graph looks like this:



This is sometimes called a **growth** function.

If $0<a<1$ then the graph looks like this:



This is sometimes called a **decay** function.

Remember that the graph of an exponential function $f(x)=a^x$ always passes through $(0,1)$ and $(1,a)$ since

$$f(0)=a^0=1, \qquad f(1)=a^1=a.$$

Exponentials and Logarithms to the Base *e*

The constant e is an important number in Mathematics, and occurs frequently in models of real-life situations. Its value is roughly 2·718281828 (to 9 d.p.), and is defined as:

$$\left(1+\frac{1}{n}\right)^n \quad \text{as } n \to \infty.$$

If you try large values of n on your calculator, you will get close to the value of e. Like π, e is an irrational number.

Throughout this section, we will use e in expressions of the form:

- e^x, which is called an exponential to the base e,
- $\log_e x$, which is called a logarithm to the base e. This is also known as the **natural logarithm** of x, and is often written as $\ln x$ (i.e. $\ln x \equiv \log_e x$).

Question 1

Evaluate a) e^2 b) $5e^{-1.2}$

Question 2

A van hire company calculates the depreciation in the value of its vans using the formula $V(t) = V_0e^{-0.16t}$ where V_0 represents the initial value. A new van costs £18000, calculate its value after 5 years.

Your question:

The population of rats is increasing according to the law $P_t = P_0e^{0.132t}$ where P_0 is the initial population and t is the time in weeks.
Given $P_0 = 500$ find the population after:

a) 2 weeks

b) 7 weeks

Logarithms

A logarithm is the inverse of an exponential.

The relationship between logarithms and exponentials is expressed as:

$$y = \log_a x \Leftrightarrow x = a^y \quad \text{where } a,\ x > 0.$$

Here, y is the power of a which gives x.

Ex 1

Write in logarithmic form

(a) $3^4 = 81$

(b) $\frac{1}{2} = 2^{-1}$

Ex 2

Simplify

(a) $\log_2 4$

(b) $\log_4 64$

(c) $\log_3 \frac{1}{27}$

Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

Laws of Logarithms

There are three laws of logarithms which you must know.

Rule 1

$$\log_a x + \log_a y = \log_a (xy) \qquad \text{where } a, x, y > 0.$$

If two logarithmic terms with the same base number (a above) are being added together, then the terms can be combined by multiplying the arguments (x and y above).

EXAMPLE

1. Simplify $\log_5 2 + \log_5 4$.

Rule 2

$$\log_a x - \log_a y = \log_a \left(\tfrac{x}{y}\right) \qquad \text{where } a, x, y > 0.$$

If a logarithmic term is being subtracted from another logarithmic term with the same base number (a above), then the terms can be combined by dividing the arguments (x and y in this case).

Note that the argument which is being taken away (y above) appears on the bottom of the fraction when the two terms are combined.

EXAMPLE

2. Evaluate $\log_4 6 - \log_4 3$.

Rule 3

$$\log_a x^n = n\log_a x \qquad \text{where } a, x > 0.$$

The power of the argument (n above) can come to the front of the term as a multiplier, and vice-versa.

EXAMPLE

3. Express $2\log_7 3$ in the form $\log_7 a$.

Note

When working with logarithms, you should remember:

$$\log_a 1 = 0 \quad \text{since } a^0 = 1, \qquad \log_a a = 1 \quad \text{since } a^1 = a.$$

EXAMPLE

4. Evaluate $\log_7 7 + \log_3 3$.

Combining several log terms

When adding and subtracting several log terms in the form $\log_a b$, there is a simple way to combine all the terms in one step.

$$\log_a\left(\frac{\text{arguments of }\textbf{positive}\text{ log terms}}{\text{arguments of }\textbf{negative}\text{ log terms}}\right)$$

- Multiply the arguments of the positive log terms in the numerator.
- Multiply the arguments of the negative log terms in the denominator.

EXAMPLES

5. Evaluate $\log_{12} 10 + \log_{12} 6 - \log_{12} 5$.

6. Evaluate $\log_6 4 + 2\log_6 3$.

Your questions

1) Simplify

(a) $\log_8 2 + \log_8 4$ (b) $3\log_3 3 + \frac{1}{2}\log_3 9$

2)

If $\log_a y = \log_a 2 + 3\log_a x$, express y in terms of x.

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1 Express in exponential form $y = \log_e 6$

2 Express in logarithmic form $y = e^7$

3 Calculate the value of $\log_e 6$

4. Solve $\log_e x = 9$

5

Simplify $4\log_e(2e) - 3\log_e(3e)$ expressing your answer in the form $a + \log_e b - \log_e c$ where a, b and c are whole numbers.

Additional examples are available in the Dynamic Maths Study Notes available at http://madrasmaths.com/courses/higher/revison_materials_higher.html (password: madrasmaths) and at hsn.uk.net

Logarithmic Equations

Ex
Solve the following equations for $x > 0$

(a) $\log_a 4 + \log_a x = \log_a 12$

(b) $\log_a (x+1) + \log_a (x-1) = \log_a 8$

Applications of Exponentials and Logarithms

The number of pairs of breeding gulls in a nature reserve is given by $P(t) = 500(1.09)^t$, where t is the time in years since records began.

(a) How many pairs were there initially?

(b) After how many years will the population exceed 2000 pairs for the first time?

A number N_0 of radioactive nuclei decay to N_t after t years according to the law $N_t = N_0 e^{-0.05t}$.

(a) Find the number remaining after 50 years if the original number N_0 was 500.

(b) The half-life of a radioactive sample is defined as the time taken for the activity to be reduced by half. Calculate the half-life for this sample.

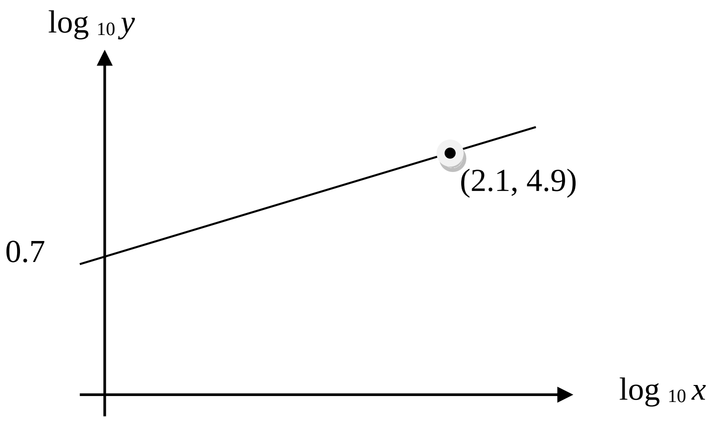
Interpreting Experimental Data

2 sets of data are often linked by exponential growth or decay. Logarithms can be applied to determine the equation of the function.

These functions can be of the form:

- $y = kx^n$
- $y = ab^x$

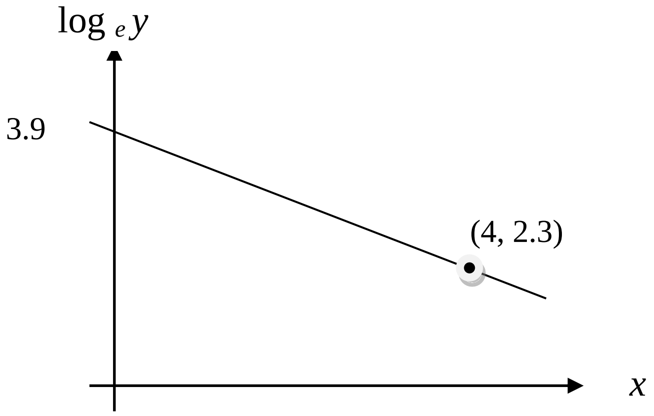
1) Results from an experiment are shown in the graph.



(a) Show this graph represents a relationship of the form $y = kx^n$

(b) Determine the values of k and n.

2) Results from an experiment are shown in the graph.



(a) Show this graph represents a relationship of the form $y = ab^x$

(b) Determine the values of a and b.

Unit Assessment Practice Questions

Logarithms and Exponentials

Practice test 1

1 (a) Simplify $\log_5 6a + \log_5 7b$.

(b) Express $\log_b x^7 - \log_b x^4$ in the form $k \log_b x$

[3]

2 Solve. $\log_4 (x-1) = 3$

[2]

Practice test 2

1 (a) Simplify $\log_4 3p - \log_4 2q$.

(b) Express $\log_a x^2 + \log_a x^3$ in the form $k \log_a x$

[3]

2 Explain why $x = 0{\cdot}399$ is a solution of the following equation to 3 significant figures:
$e^{5x+1} = 20$

[2 + #2.2]

Higher Maths Homework – Logarithms and Exponentials

Attempt all questions. Do not leave any blanks!

Video help is available at YouTube.com/DLBMaths or search for e.g. YouTube DLBMaths SQA Higher Maths 2012 Question 7

Once you have **completed and marked** your homework using the videos above please grade yourself on each question using the following code:

1 – fully understood and completed on own

2 – partially understood and now understand after using video help

3 – looked at video help, copied down the solution but will need extra help from my teacher.

Non Calculator Section

1

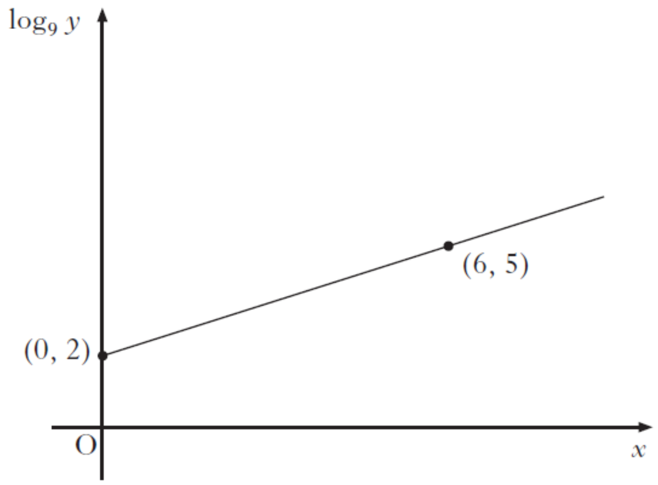
Evaluate $\log_6 12 + \frac{1}{3}\log_6 27$. 3

(SQA (New) Higher Maths 2015 Paper 1 Question 6)

2 Two variables, x and y, are related by the equation

$$y = ka^x.$$

When $\log_9 y$ is plotted against x, a straight line passing through the points (0, 2) and (6, 5) is obtained, as shown in the diagram.



5

Find the values of k and a.

(SQA Higher Maths 2014 Paper 1 Question 24)

Calculator Section

3 Solve the equation

$$\log_5(3-2x) + \log_5(2+x) = 1, \text{ where } x \text{ is a real number.}$$ 4

(SQA Higher Maths 2013 Paper 2 Question 5)

4 The concentration of the pesticide, *Xpesto*, in soil can be modelled by the equation

$$P_t = P_0 e^{-kt}$$

where:

- P_0 is the initial concentration;
- P_t is the concentration at time t;
- t is the time, in days, after the application of the pesticide.

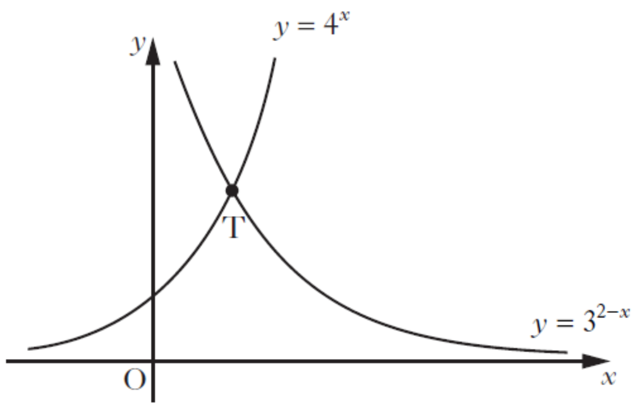
(*a*) Once in the soil, the half-life of a pesticide is the time taken for its concentration to be reduced to one half of its initial value. 4

If the half-life of *Xpesto* is 25 days, find the value of k to 2 significant figures.

(*b*) Eighty days after the initial application, what is the percentage decrease in concentration of *Xpesto*? 3

(SQA Higher Maths 2013 Paper 2 Question 9)

5 The diagram shows the curves with equations $y = 4^x$ and $y = 3^{2-x}$.



The graphs intersect at the point T.

(*a*) Show that the x – coordinate of T can be written in the form $\dfrac{\log_a p}{\log_a q}$, for all $a > 1$. 6

(*b*) Calculate the y – coordinate of T. 2

(SQA Higher Maths 2012 Paper 2 Question 7)

Unit Assessment Practice Solutions

Practice test 1

1 (a) $\log_5 6a + \log_5 7b$

$= \log_5 (6a \times 7b)$

$= \log_5 42ab$

(b) $\log_6 x^7 - \log_6 x^4$

$= 7\log_6 x - 4\log_6 x$

$= 3\log_6 x$

or $\log_6 x^7 - \log_6 x^4$

$= \log_6 \frac{x^7}{x^4}$

$= \log_6 x^3$

$= 3\log_6 x$

2 $\log_4 (x-1) = 3$

$x - 1 = 4^3$

$x - 1 = 64$

$x = 65$

Practice test 2

1(a) $\log_4 3p - \log_4 2q$

$= \log_4 \frac{3p}{2q}$

(b) $2\log_a x + 3\log_a x$

$= 5\log_a x$

2 $e^{5x+1} = 20$

$\ln e^{5x+1} = \ln 20$

$(5x+1)\cancel{\ln e}^{1} = \ln 20$

$5x = \ln(20) - 1$

$x = \frac{\ln(20) - 1}{5}$

$x = 0.399146...$

$x = 0.399$ (3 s.f.)